

Blow-up criteria for Boussinesq system and MHD system and Landau-Lifshitz equations in a bounded domain

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Abstract

In this paper, we prove some blow-up criteria for the 3D Boussinesq system with zero heat conductivity and MHD system and Landau-Lifshitz equations in a bounded domain.

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1 Introduction

Let Ω be a bounded, simply connected domain in \mathbb{R}^3 with smooth boundary $\partial\Omega$, and ν be the unit outward normal vector to $\partial\Omega$. First, we consider the regularity criterion of the

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Boussinesq system with zero heat conductivity:

$$\operatorname{div} u = 0, \quad (1.1)$$

$$\partial_t u + u \cdot \nabla u + \nabla \pi - \Delta u = \theta e_3, \quad (1.2)$$

$$\partial_t \theta + u \cdot \nabla \theta = 0 \quad \text{in } \Omega \times (0, \infty), \quad (1.3)$$

$$u \cdot \nu = 0, \operatorname{curl} u \times \nu = 0 \quad \text{on } \partial\Omega \times (0, \infty), \quad (1.4)$$

$$(u, \theta)(\cdot, 0) = (u_0, \theta_0) \quad \text{in } \Omega \subseteq \mathbb{R}^3, \quad (1.5)$$

where u, π , and θ denote unknown velocity vector field, pressure scalar, and temperature scalar of the fluid, respectively. $\omega := \operatorname{curl} u$ is the vorticity and $e_3 := (0, 0, 1)^t$.

When $\theta = 0$, (1.1) and (1.2) are the well-known Navier-Stokes system. Giga [1], Kim [2], Kang and Kim [3] have proved some Serrin type regularity criteria.

The first aim of this paper is to prove a new regularity criterion for the problem (1.1)-(1.5), we will prove

Theorem 1.1. *Let $u_0 \in H^3, \theta_0 \in W^{1,p}$ with $3 < p \leq 6$ and $\operatorname{div} u_0 = 0$ in Ω and $u_0 \cdot \nu = 0, \operatorname{curl} u_0 \times \nu = 0$ on $\partial\Omega$. Let (u, θ) be a strong solution of the problem (1.1)-(1.5). If u satisfies*

$$\nabla u \in L^1(0, T; BMO(\Omega)) \quad (1.6)$$

with $0 < T < \infty$, then the solution (u, θ) can be extended beyond $T > 0$. Here BMO denotes the space of bounded mean oscillation.

Secondly, we consider the blow-up criterion of the 3D MHD system

$$\operatorname{div} u = \operatorname{div} b = 0, \quad (1.7)$$

$$\partial_t u + u \cdot \nabla u + \nabla \left(\pi + \frac{1}{2} |b|^2 \right) - \Delta u = b \cdot \nabla b, \quad (1.8)$$

$$\partial_t b + u \cdot \nabla b - b \cdot \nabla u = \Delta b \quad \text{in } \Omega \times (0, \infty), \quad (1.9)$$

$$u \cdot \nu = 0, \operatorname{curl} u \times \nu = 0, b \cdot \nu = 0, \operatorname{curl} b \times \nu = 0 \quad \text{on } \partial\Omega \times (0, \infty), \quad (1.10)$$

$$(u, b)(\cdot, 0) = (u_0, b_0) \quad \text{in } \Omega \subseteq \mathbb{R}^3. \quad (1.11)$$

Here b is the magnetic field of the fluid.

It is well-known that the problem (1.7)-(1.11) has a unique local strong solution [4]. But whether this local solution can exist globally is an outstanding problem. Kang and Kim [3] prove some Serrin type regularity criteria.

The second aim of this paper is to prove a new regularity criterion for the problem (1.7)-(1.11), we will prove

Theorem 1.2. *Let $u_0, b_0 \in H^3$ with $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$ in Ω and $u_0 \cdot \nu = b_0 \cdot \nu = 0, \operatorname{curl} u_0 \times \nu = \operatorname{curl} b_0 \times \nu = 0$ on $\partial\Omega$. Let (u, b) be a strong solution to the problem (1.7)-(1.11). If (1.6) holds true, then the solution (u, b) can be extended beyond $T > 0$.*

Remark 1.1. When $\Omega := \mathbb{R}^3$, our result gives the following well-known regularity criterion

$$\omega := \operatorname{curl} u \in L^1(0, T; \dot{B}_{\infty, \infty}^0),$$

but the method of proof we used is different from that in [14, 15]. Here $\dot{B}_{\infty, \infty}^0$ denotes the homogeneous Besov space [13].

Next, we consider the following 3D density-dependent MHD equations:

$$\operatorname{div} u = \operatorname{div} b = 0, \quad (1.12)$$

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \quad (1.13)$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla \left(\pi + \frac{1}{2}|b|^2 \right) - \Delta u = b \cdot \nabla b, \quad (1.14)$$

$$\partial_t b + u \cdot \nabla b - b \cdot \nabla u = \Delta b \quad \text{in } \Omega \times (0, \infty), \quad (1.15)$$

$$u = 0, b \cdot \nu = 0, \operatorname{curl} b \times \nu = 0 \quad \text{on } \partial\Omega \times (0, \infty), \quad (1.16)$$

$$(\rho, \rho u, b)(\cdot, 0) = (\rho_0, \rho_0 u_0, b_0) \quad \text{in } \Omega \subset \mathbb{R}^3. \quad (1.17)$$

For this problem, in [5], Wu proved that if the initial data ρ_0, u_0 , and b_0 satisfy

$$0 \leq \rho_0 \in H^2, u_0 \in H_0^1 \cap H^2, b_0 \in H^2, -\Delta u_0 + \nabla \left(\pi_0 + \frac{1}{2}|b_0|^2 \right) = b_0 \cdot \nabla b_0 + \sqrt{\rho_0} g \quad (1.18)$$

for some $(\pi_0, g) \in H^1 \times L^2$, then there exists a positive time T_* and a unique strong solution (ρ, u, b) to the problem (1.12)-(1.17) such that

$$\begin{aligned} \rho &\in C([0, T_*]; H^2), u \in C([0, T_*]; H_0^1 \cap H^2) \cap L^2(0, T_*; H^2), \\ u_t &\in L^2(0, T_*; H_0^1), \sqrt{\rho} u_t \in L^\infty(0, T_*; L^2), \\ b &\in L^\infty(0, T_*; H^2) \cap L^2(0, T_*; H^3), b_t \in L^\infty(0, T_*; L^2) \cap L^2(0, T_*; H^1). \end{aligned} \quad (1.19)$$

And when $b = 0$, Kim [2] proved the following regularity criterion:

$$u \in L^{\frac{2s}{s-3}}(0, T; L_w^s(\Omega)) \quad \text{with } 3 < s \leq \infty. \quad (1.20)$$

Here L_w^s denotes the weak- L^s space and $L_w^\infty = L^\infty$.

The aim of this paper is to refine (1.20), we will prove

Theorem 1.3. *Let ρ_0, u_0 , and b_0 satisfy (1.18). Let (ρ, u, b) be a strong solution of the problem (1.12)-(1.17) in the class (1.19). If u satisfies one of the following two conditions:*

$$(i) \quad \int_0^T \frac{\|u(t)\|_{L_w^{\frac{2s}{s-3}}}^{\frac{2s}{s-3}}}{1 + \log(e + \|u(t)\|_{L_w^s})} dt < \infty \quad \text{with } 3 < s \leq \infty, \quad (1.21)$$

$$(ii) \quad u \in L^2(0, T; BMO(\Omega)) \quad (1.22)$$

with $0 < T < \infty$, then the solution (ρ, u, b) can be extended beyond $T > 0$.

Finally, we consider the 3D Landau-Lifshitz system:

$$\partial_t d - \Delta d = d|\nabla d|^2 + d \times \Delta d, |d| = 1 \text{ in } \Omega \times (0, \infty), \quad (1.23)$$

$$\partial_\nu d = 0 \text{ on } \partial\Omega \times (0, \infty), \quad (1.24)$$

$$d(\cdot, 0) = d_0, |d_0| = 1 \text{ in } \Omega \subseteq \mathbb{R}^3. \quad (1.25)$$

Carbou and Fabrie [6] showed the existence and uniqueness of local smooth solutions. When $\Omega := \mathbb{R}^n$ ($n = 2, 3, 4$), Fan and Ozawa [7] proved some regularity criteria. The aim of this paper is to prove a logarithmic blow-up criterion for the problem (1.23)-(1.25) when Ω is a bounded domain. We will prove

Theorem 1.4. *Let $d_0 \in H^3(\Omega)$ with $|d_0| = 1$ in Ω and $\partial_\nu d_0 = 0$ on $\partial\Omega$. Let d be a local smooth solution to the problem (1.23)-(1.25). If d satisfies*

$$\int_0^T \frac{\|\nabla d\|_{L^q}^{\frac{2q}{q-3}}}{1 + \log(e + \|\nabla d\|_{L^q})} dt < \infty \text{ with } 3 < q \leq \infty, \quad (1.26)$$

and $0 < T < \infty$, then the solution can be extended beyond $T > 0$.

In the following section 2, we give some preliminary Lemmas which will be used in the following sections. The proof of Theorem 1.1 of problem (1.1) -(1.5) will be given in section 3. The new regularity criterion of Theorem 1.2 for the 3D MHD problem (1.7) -(1.11) will be proved in section 4. In section 5 is the proof of the Theorem 1.3, and in the next section 6 we give the main proof of final Theorem 1.4.

2 Preliminary Lemmas

In the following proofs, we will use the following logarithmic Sobolev inequality [8]:

$$\|\nabla u\|_{L^\infty} \leq C(1 + \|\nabla u\|_{BMO} \log(e + \|u\|_{W^{s,p}})) \text{ with } s > 1 + \frac{3}{p}. \quad (2.1)$$

and the following three lemmas.

Lemma 2.1. ([9]). *Let $\Omega \subseteq \mathbb{R}^3$ be a smooth bounded domain, let $b : \Omega \rightarrow \mathbb{R}^3$ be a smooth vector field, and let $1 < p < \infty$. Then*

$$\begin{aligned} - \int_\Omega \Delta b \cdot b |b|^{p-2} dx &= \frac{1}{2} \int_\Omega |b|^{p-2} |\nabla b|^2 dx + 4 \frac{p-2}{p^2} \int_\Omega |\nabla |b|^{\frac{p}{2}}|^2 dx \\ &\quad - \int_{\partial\Omega} |b|^{p-2} (b \cdot \nabla) b \cdot \nu d\sigma - \int_{\partial\Omega} |b|^{p-2} (\text{curl } b \times \nu) \cdot b d\sigma. \end{aligned} \quad (2.2)$$

Lemma 2.2. ([10, 11]). *Let Ω be a smooth and bounded open set and let $1 < p < \infty$. Then the following estimate:*

$$\|b\|_{L^p(\partial\Omega)} \leq C \|b\|_{L^p(\Omega)}^{1-\frac{1}{p}} \|b\|_{W^{1,p}(\Omega)}^{\frac{1}{p}} \quad (2.3)$$

holds for any $b \in W^{1,p}(\Omega)$.

Lemma 2.3. *There holds*

$$\|f\|_{L^\infty(\Omega)} \leq C(1 + \|f\|_{BMO(\Omega)} \log^{\frac{1}{2}}(e + \|f\|_{W^{1,4}(\Omega)})) \quad (2.4)$$

for any $f \in W_0^{1,4}(\Omega)$.

Proof. When $\Omega := \mathbb{R}^3$, (2.4) has been proved in Ogawa [12]. When Ω is a bounded domain in \mathbb{R}^3 . We can define

$$\tilde{f} := \begin{cases} f & \text{in } \Omega, \\ 0 & \text{in } \Omega^c := \mathbb{R}^3 \setminus \Omega. \end{cases}$$

Then we have [10, Page 71]

$$\|\tilde{f}\|_{W^{1,4}(\mathbb{R}^3)} = \|f\|_{W^{1,4}(\Omega)}$$

and it is obvious that

$$\|\tilde{f}\|_{L^\infty(\mathbb{R}^3)} = \|f\|_{L^\infty(\Omega)}, \|\tilde{f}\|_{BMO(\mathbb{R}^3)} = \|f\|_{BMO(\Omega)}.$$

Thus (2.4) is proved. □

Finally, when b satisfies $b \cdot \nu = 0$ on $\partial\Omega$, we will also use the identity

$$(b \cdot \nabla)b \cdot \nu = -(b \cdot \nabla)\nu \cdot b \quad \text{on } \partial\Omega \quad (2.5)$$

for any sufficiently smooth vector field b .

3 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. Since it is easy to prove that the problem (1.1) -(1.5) has a unique local-in-time strong solution, we omit the details here. We only need to establish a priori estimates.

First, thanks to the maximum principle, it follows from (1.1) and (1.3) that

$$\|\theta\|_{L^\infty(0,T;L^\infty)} \leq C. \quad (3.1)$$

Testing (1.2) by u and using (1.1) and (3.1), we see that

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 dx + \int_{\Omega} |\operatorname{curl} u|^2 dx \leq \int_{\Omega} \theta e_3 \cdot u dx \leq \frac{1}{2} \int_{\Omega} \theta^2 dx + \frac{1}{2} \int_{\Omega} u^2 dx,$$

which gives

$$\|u\|_{L^\infty(0,T;L^2)} + \|u\|_{L^2(0,T;H^1)} \leq C. \quad (3.2)$$

Applying curl to (1.2) and setting $\omega := \operatorname{curl} u$, we find that

$$\partial_t \omega + u \cdot \nabla \omega - \Delta \omega = \omega \cdot \nabla u + \operatorname{curl}(\theta e_3). \quad (3.3)$$

Testing (3.3) by ω and using (1.1) and (3.1), we infer that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\omega|^2 dx + \int_{\Omega} |\operatorname{curl} \omega|^2 dx &= \int_{\Omega} (\omega \cdot \nabla) u \cdot \omega dx + \int_{\Omega} \theta e_3 \operatorname{curl} \omega dx \\ &\leq \|\nabla u\|_{L^\infty} \int_{\Omega} \omega^2 dx + \frac{1}{2} \int_{\Omega} |\operatorname{curl} \omega|^2 dx + C, \end{aligned}$$

which implies

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} |\omega|^2 dx + \int_{\Omega} |\operatorname{curl} \omega|^2 dx &\leq C \|\nabla u\|_{L^\infty} \int_{\Omega} |\omega|^2 dx + C \\ &\leq C(1 + \|\nabla u\|_{BMO}) \log(e + \|u\|_{H^3}) \int_{\Omega} |\omega|^2 dx + C, \end{aligned}$$

and therefore

$$\int_{\Omega} |\omega|^2 dx + \int_{t_0}^t \|\operatorname{curl} \omega\|_{L^2}^2 d\tau \leq C(e + y)^{C_0 \epsilon} \quad (3.4)$$

provided that

$$\int_{t_0}^t \|\nabla u\|_{BMO} d\tau \leq \epsilon \ll 1 \quad (3.5)$$

and $y(t) := \sup_{[t_0, t]} \|u\|_{H^3}$ for any $0 < t_0 \leq t \leq T$ and C_0 is an absolute constant.

Applying ∂_t to (1.2), we deduce that

$$\partial_t^2 u + u \cdot \nabla u_t + \nabla \pi_t - \Delta u_t = -u_t \cdot \nabla u + \theta_t e_3. \quad (3.6)$$

Testing (3.6) by u_t , using (1.1), (1.3), (3.1) and (3.2), we derive

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u_t|^2 dx + \int_{\Omega} |\operatorname{curl} u_t|^2 dx \\ &= - \int_{\Omega} u_t \cdot \nabla u \cdot u_t dx + \int_{\Omega} \theta_t e_3 u_t dx \\ &= - \int_{\Omega} u_t \cdot \nabla u \cdot u_t dx - \int_{\Omega} \operatorname{div}(u \theta) e_3 u_t dx \\ &= - \int_{\Omega} u_t \cdot \nabla u \cdot u_t dx + \int_{\Omega} u \theta \nabla(e_3 u_t) dx \\ &\leq \|\nabla u\|_{L^\infty} \int_{\Omega} |u_t|^2 dx + \frac{1}{2} \int_{\Omega} |\operatorname{curl} u_t|^2 dx + C \\ &\leq C(1 + \|\nabla u\|_{BMO}) \log(e + y) \int_{\Omega} |u_t|^2 dx + \frac{1}{2} \int_{\Omega} |\operatorname{curl} u_t|^2 dx + C, \end{aligned}$$

which yields

$$\int_{\Omega} |u_t|^2 dx + \int_{t_0}^t \int_{\Omega} |\operatorname{curl} u_t|^2 dx d\tau \leq C(e + y)^{C_0 \epsilon}. \quad (3.7)$$

On the other hand, thanks to the H^2 -theory of the Stokes system, it follows from (1.2), (3.1), (3.4) and (3.7) that

$$\begin{aligned}
\|u\|_{H^2} &\leq C\|-\Delta u + \nabla \pi\|_{L^2} \\
&\leq C\|\partial_t u + u \cdot \nabla u - \theta e_3\|_{L^2} \\
&\leq C\|u_t\|_{L^2} + C\|u\|_{L^6}\|\nabla u\|_{L^3} + C\|\theta\|_{L^2} \\
&\leq C\|u_t\|_{L^2} + C\|\nabla u\|_{L^2}^{\frac{3}{2}}\|u\|_{H^2}^{\frac{1}{2}} + C,
\end{aligned}$$

which implies

$$\|u\|_{H^2} \leq C\|u_t\|_{L^2} + C\|\nabla u\|_{L^2}^3 + C \leq C(e+y)^{C_0\epsilon}. \quad (3.8)$$

Applying ∇ to (1.3), testing by $|\nabla \theta|^{p-2} \nabla \theta$ ($2 \leq p < \infty$) and using (1.1), we get

$$\begin{aligned}
\frac{d}{dt} \|\nabla \theta\|_{L^p} &\leq C\|\nabla u\|_{L^\infty} \|\nabla \theta\|_{L^p} \\
&\leq C(1 + \|\nabla u\|_{BMO}) \log(e+y) \|\nabla \theta\|_{L^p},
\end{aligned}$$

which leads to

$$\|\nabla \theta\|_{L^\infty(t_0, t; L^p)} \leq C(e+y)^{C_0\epsilon} \quad \text{with } 2 \leq p < \infty. \quad (3.9)$$

Testing (3.6) by $-\Delta u_t + \nabla \pi_t$, using (1.1), (1.3), (3.7), (3.8) and (3.9), we obtain

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\operatorname{curl} u_t|^2 dx + \int_{\Omega} |-\Delta u_t + \nabla \pi_t|^2 dx \\
&= \int_{\Omega} (-u_t \cdot \nabla u + \theta_t e_3 - u \cdot \nabla u_t) (-\Delta u_t + \nabla \pi_t) dx \\
&\leq (\|\nabla u\|_{L^6} \|u_t\|_{L^3} + \|u\|_{L^\infty} \|\nabla \theta\|_{L^2} + \|u\|_{L^\infty} \|\nabla u_t\|_{L^2}) \|-\Delta u_t + \nabla \pi_t\|_{L^2} \\
&\leq \|u\|_{H^2} (\|u_t\|_{H^1} + \|\nabla \theta\|_{L^2}) \|-\Delta u_t + \nabla \pi_t\|_{L^2} \\
&\leq \frac{1}{2} \|-\Delta u_t + \nabla \pi_t\|_{L^2}^2 + C\|u\|_{H^2}^2 (\|u_t\|_{H^1}^2 + \|\nabla \theta\|_{L^2}^2),
\end{aligned}$$

which leads to

$$\int_{\Omega} |\operatorname{curl} u_t|^2 dx + \int_{t_0}^t \|u_t\|_{H^2}^2 d\tau \leq C(e+y)^{C_0\epsilon}. \quad (3.10)$$

On the other hand, it follows from (3.3), (3.10), (3.9) and (3.8) that

$$\begin{aligned}
\|u\|_{H^3} &\leq C(1 + \|\Delta \omega\|_{L^2}) \\
&\leq C(1 + \|\partial_t \omega + u \cdot \nabla \omega - \omega \cdot \nabla u - \operatorname{curl}(\theta e_3)\|_{L^2}) \\
&\leq C(1 + \|\partial_t \omega\|_{L^2} + \|u\|_{L^\infty} \|\nabla \omega\|_{L^2} + \|\omega\|_{L^4} \|\nabla u\|_{L^4} + \|\nabla \theta\|_{L^2}) \\
&\leq C(e+y)^{C_0\epsilon},
\end{aligned}$$

which gives

$$\|u\|_{L^\infty(0, T; H^3)} \leq C, \quad (3.11)$$

and

$$\|\theta\|_{L^\infty(0, T; W^{1, p})} \leq C \quad \text{with } 3 \leq p \leq 6. \quad (3.12)$$

This completes the proof of Theorem 1.1. \square

4 Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2, we only need to prove a priori estimates.

First, testing (1.8) by u and using (1.7), we see that

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 dx + \int_{\Omega} |\operatorname{curl} u|^2 dx = \int_{\Omega} (b \cdot \nabla) b \cdot u dx. \quad (4.1)$$

Testing (1.9) by b and using (1.7), we find that

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} b^2 dx + \int_{\Omega} |\operatorname{curl} b|^2 dx = \int_{\Omega} (b \cdot \nabla) u \cdot b dx. \quad (4.2)$$

Summing up (4.1) and (4.2), we get the well-known energy inequality

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} (u^2 + b^2) dx + \int_{\Omega} (|\operatorname{curl} u|^2 + |\operatorname{curl} b|^2) dx \leq 0. \quad (4.3)$$

Testing (1.9) by $|b|^{p-2}b$ ($2 \leq p \leq 6$), using (1.7), (2.2), (2.3) and (2.5), we derive

$$\begin{aligned} & \frac{1}{p} \frac{d}{dt} \int_{\Omega} |b|^p dx + \frac{1}{2} \int_{\Omega} |b|^{p-2} |\nabla b|^2 dx + 4 \frac{p-2}{p^2} \int_{\Omega} |\nabla |b|^{\frac{p}{2}}|^2 dx \\ &= - \int_{\partial\Omega} |b|^{p-2} (b \cdot \nabla) \nu \cdot b d\sigma + \int_{\Omega} b \cdot \nabla u \cdot |b|^{p-2} b dx \\ &\leq C \int_{\partial\Omega} |b|^p dx + \|\nabla u\|_{L^\infty} \int_{\Omega} |b|^p dx \\ &\leq 2 \frac{p-2}{p^2} \int_{\Omega} |\nabla |b|^{\frac{p}{2}}|^2 dx + C(1 + \|\nabla u\|_{L^\infty}) \int_{\Omega} |b|^p dx \\ &\leq 2 \frac{p-2}{p^2} \int_{\Omega} |\nabla |b|^{\frac{p}{2}}|^2 dx + C(1 + \|\nabla u\|_{BMO}) \int_{\Omega} |b|^p dx \log(e + y), \end{aligned}$$

which implies

$$\|b\|_{L^\infty(t_0, t; L^p)} + \int_{t_0}^t \int_{\Omega} |b|^2 |\nabla b|^2 dx d\tau \leq C(e + y)^{C_0 \epsilon} \quad \text{with } 2 \leq p \leq 6, \quad (4.4)$$

with the same y and ϵ as that in (3.5).

Taking curl to (1.8) and (1.9), respectively, and setting $\omega := \operatorname{curl} u$ and $j := \operatorname{curl} b$, we infer that

$$\partial_t \omega + u \cdot \nabla \omega - \Delta \omega = \omega \cdot \nabla u + b \cdot \nabla j + \sum_i \nabla b_i \times \partial_i b, \quad (4.5)$$

$$\partial_t j + u \cdot \nabla j - \Delta j = b \cdot \nabla \omega + \sum_i \nabla b_i \times \partial_i u - \sum_i \nabla u_i \times \partial_i b. \quad (4.6)$$

Testing (4.5) and (4.6) by ω and j , respectively, summing up the result and using (1.7), we have

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_{\Omega} (\omega^2 + j^2) dx + \int_{\Omega} (|\operatorname{curl} \omega|^2 + |\operatorname{curl} j|^2) dx \\
&= \int_{\Omega} (\omega \cdot \nabla) u \cdot \omega dx + \sum_i \int_{\Omega} (\nabla b_i \times \partial_i b) \omega dx + \sum_i \int_{\Omega} (\nabla b_i \times \partial_i u) \cdot j dx - \sum_i \int_{\Omega} (\nabla u_i \times \partial_i b) \cdot j dx \\
&\leq C \|\nabla u\|_{L^\infty} \int_{\Omega} (\omega^2 + j^2) dx \\
&\leq C(1 + \|\nabla u\|_{BMO}) \int_{\Omega} (\omega^2 + j^2) dx \log(e + y),
\end{aligned}$$

which implies

$$\int_{\Omega} (\omega^2 + j^2) dx + \int_{t_0}^t \int_{\Omega} (|\operatorname{curl} \omega|^2 + |\operatorname{curl} j|^2) dx d\tau \leq C(e + y)^{C_0 \epsilon}. \quad (4.7)$$

Thus, it follows from (1.8), (1.9) and (4.7) that

$$\int_{t_0}^t \int_{\Omega} (|u_t|^2 + |b_t|^2) dx d\tau \leq C(e + y)^{C_0 \epsilon}. \quad (4.8)$$

Applying ∂_t to (1.8), we have

$$\partial_t^2 u + u \cdot \nabla u_t + \nabla \pi_t - \Delta u_t = \operatorname{div} (b \otimes b)_t - u_t \cdot \nabla u. \quad (4.9)$$

Testing (4.9) by u_t and using (1.7), we get

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_{\Omega} |u_t|^2 dx + \int_{\Omega} |\operatorname{curl} u_t|^2 dx \\
&= - \sum_{i,j} \int_{\Omega} (b^i b^j)_t \partial_j u_t^i dx - \int_{\Omega} u_t \cdot \nabla u \cdot u_t dx \\
&\leq C \|b_t\|_{L^3} \|b\|_{L^6} \|\nabla u_t\|_{L^2} + \|\nabla u\|_{L^2} \|u_t\|_{L^4}^2 \\
&\leq C \|b_t\|_{L^2}^{\frac{1}{2}} \|\operatorname{curl} b_t\|_{L^2}^{\frac{1}{2}} \|\operatorname{curl} u_t\|_{L^2} \|b\|_{L^6} + C \|\nabla u\|_{L^2} \|u_t\|_{L^2}^{\frac{1}{2}} \|\operatorname{curl} u_t\|_{L^2}^{\frac{3}{2}} \\
&\leq \delta \|\operatorname{curl} u_t\|_{L^2}^2 + \delta \|\operatorname{curl} b_t\|_{L^2}^2 + C \|b_t\|_{L^2}^2 \|b\|_{L^6}^4 + C \|\nabla u\|_{L^2}^4 \|u_t\|_{L^2}^2
\end{aligned} \quad (4.10)$$

for any $\delta \in (0, 1)$.

Applying ∂_t to (1.9), we have

$$\partial_t^2 b + u \cdot \nabla b_t - \Delta b_t = b_t \cdot \nabla u + b \cdot \nabla u_t - u_t \cdot \nabla b. \quad (4.11)$$

Testing (4.11) by b_t and using (1.7), we deduce that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_{\Omega} |b_t|^2 dx + \int_{\Omega} |\operatorname{curl} b_t|^2 dx \\
&= \int_{\Omega} (b_t \cdot \nabla u + b \cdot \nabla u_t - u_t \cdot \nabla b) b_t dx \\
&\leq \|\nabla u\|_{L^2} \|b_t\|_{L^4}^2 + \|b\|_{L^6} \|\nabla u_t\|_{L^2} \|b_t\|_{L^3} + \|\nabla b\|_{L^2} \|u_t\|_{L^4} \|b_t\|_{L^4} \\
&\leq \delta \|\operatorname{curl} b_t\|_{L^2}^2 + \delta \|\operatorname{curl} u_t\|_{L^2}^2 \\
&\quad + C \|\nabla u\|_{L^2}^4 \|b_t\|_{L^2}^2 + C \|b\|_{L^6}^4 \|b_t\|_{L^2}^2 + C \|\nabla b\|_{L^2}^4 (\|u_t\|_{L^2}^2 + \|b_t\|_{L^2}^2)
\end{aligned} \tag{4.12}$$

for any $\delta \in (0, 1)$.

Combining (4.10) and (4.12) and taking δ small enough and using (4.7) and (4.8), we have

$$\int_{\Omega} (|u_t|^2 + |b_t|^2) dx + \int_{t_0}^t \int_{\Omega} (|\operatorname{curl} u_t|^2 + |\operatorname{curl} b_t|^2) dx d\tau \leq C(e + y)^{C_0\epsilon}. \tag{4.13}$$

It follows from (1.8), (1.9), (4.7) and (4.13) that

$$\|u\|_{L^\infty(t_0, t; H^2)} + \|b\|_{L^\infty(t_0, t; H^2)} \leq C(e + y)^{C_0\epsilon}. \tag{4.14}$$

Testing (4.9) by $\nabla \left(\pi + \frac{1}{2} |b|^2 \right)_t - \Delta u_t$, and using (1.7), we find that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\operatorname{curl} u_t|^2 dx + \int_{\Omega} \left| \nabla \left(\pi + \frac{1}{2} |b|^2 \right)_t - \Delta u_t \right|^2 dx \\
&= \int_{\Omega} ((b \cdot \nabla b)_t - u_t \cdot \nabla u - u \cdot \nabla u_t) \left(\nabla \left(\pi + \frac{1}{2} |b|^2 \right)_t - \Delta u_t \right) dx \\
&\leq C(\|b\|_{L^\infty} \|\nabla b_t\|_{L^2} + \|b_t\|_{L^6} \|\nabla b\|_{L^3} + \|u_t\|_{L^6} \|\nabla u\|_{L^3} \\
&\quad + \|u\|_{L^\infty} \|\nabla u_t\|_{L^2}) \left\| \nabla \left(\pi + \frac{1}{2} |b|^2 \right)_t - \Delta u_t \right\|_{L^2} \\
&\leq \frac{1}{4} \left\| \nabla \left(\pi + \frac{1}{2} |b|^2 \right)_t - \Delta u_t \right\|_{L^2}^2 + C(\|u\|_{L^\infty}^2 + \|\nabla u\|_{L^3}^2) \|\nabla u_t\|_{L^2}^2 \\
&\quad + C(\|b\|_{L^\infty}^2 + \|\nabla b\|_{L^3}^2) \|\nabla b_t\|_{L^2}^2.
\end{aligned} \tag{4.15}$$

Similarly, testing (4.11) by $-\Delta b_t$, we infer that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\operatorname{curl} b_t|^2 dx + \int_{\Omega} |\Delta b_t|^2 dx \\
&= \int_{\Omega} (u_t \cdot \nabla b + u \cdot \nabla b_t - b_t \cdot \nabla u - b \cdot \nabla u_t) \Delta b_t dx \\
&\leq (\|u_t\|_{L^6} \|\nabla b\|_{L^3} + \|u\|_{L^\infty} \|\nabla b_t\|_{L^2} + \|\nabla u\|_{L^3} \|b_t\|_{L^6} + \|b\|_{L^\infty} \|\nabla u_t\|_{L^2}) \|\Delta b_t\|_{L^2} \\
&\leq \frac{1}{4} \|\Delta b_t\|_{L^2}^2 + C(\|u\|_{L^\infty}^2 + \|\nabla u\|_{L^3}^2) \|\nabla b_t\|_{L^2}^2 + C(\|b\|_{L^\infty}^2 + \|\nabla b\|_{L^3}^2) \|\nabla u_t\|_{L^2}^2.
\end{aligned} \tag{4.16}$$

Combining (4.15) and (4.16) and using (4.14) and (4.13), we have

$$\int_{\Omega} (|\operatorname{curl} u_t|^2 + |\operatorname{curl} b_t|^2) dx + \int_{t_0}^t \int_{\Omega} (|\Delta u_t|^2 + |\Delta b_t|^2) dx d\tau \leq C(e + y)^{C_0\epsilon}. \quad (4.17)$$

On the other hand, it follows from (4.5), (4.6), (4.3), (4.17) and (4.14) that

$$\begin{aligned} & \|u(t)\|_{H^3} + \|b(t)\|_{H^3} \leq C(1 + \|\Delta\omega\|_{L^2} + \|\Delta j\|_{L^2}) \\ & \leq C(1 + \|\partial_t\omega + u \cdot \nabla\omega - \omega \cdot \nabla u - b \cdot \nabla j - \sum_i \nabla b_i \times \partial_i b\|_{L^2} \\ & \quad + \|\partial_t j + u \cdot \nabla j - b \cdot \nabla\omega + \sum_i \nabla u_i \times \partial_i b - \sum_i \nabla b_i \times \partial_i u\|_{L^2}) \\ & \leq C(e + y(t))^{C_0\epsilon}, \end{aligned}$$

which yields

$$\|u\|_{L^\infty(0,T;H^3)} + \|b\|_{L^\infty(0,T;H^3)} \leq C,$$

This completes the proof of Theorem 1.2. □

5 Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3, we only need to establish a priori estimates.

First, it follows from (1.12) and (1.13) that

$$\|\rho\|_{L^\infty(0,T;L^\infty)} \leq C. \quad (5.1)$$

Testing (1.14) by u and using (1.12) and (1.13), we see that

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho u^2 dx + \int_{\Omega} |\nabla u|^2 dx = \int_{\Omega} (b \cdot \nabla) b \cdot u dx. \quad (5.2)$$

And testing (1.15) by b and using (1.12) and (1.16), we find that

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |b|^2 dx + \int_{\Omega} |\operatorname{curl} b|^2 dx = \int_{\Omega} (b \cdot \nabla) u \cdot b dx. \quad (5.3)$$

Summing up (5.2) and (5.3), we get the well-known energy inequality

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} (\rho |u|^2 + |b|^2) dx + \int_{\Omega} (|\nabla u|^2 + |\operatorname{curl} b|^2) dx \leq 0. \quad (5.4)$$

(I) Let (1.21) hold true.

Testing (1.15) by $|b|^{p-2}b$ ($2 \leq p < \infty$), using (1.12), (2.2), (2.3) and (2.5), and setting $\phi = |b|^{\frac{p}{2}}$, and using the Gagliardo-Nirenberg inequality [3]:

$$\|\phi\|_{L^{\frac{2s}{s-2},2}} \leq C \|\phi\|_{L^2}^{1-\frac{3}{s}} \|\phi\|_{H^1}^{\frac{3}{s}} \quad \text{with } 3 < s \leq \infty, \quad (5.5)$$

and the generalized Hölder inequality [13]:

$$\|fg\|_{L^{p,q}} \leq C \|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}} \quad (5.6)$$

with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, we derive

$$\begin{aligned} & \frac{1}{p} \frac{d}{dt} \int_{\Omega} |b|^p dx + \frac{1}{2} \int_{\Omega} |b|^{p-2} |\nabla b|^2 dx + 4 \frac{p-2}{p^2} \int_{\Omega} |\nabla |b|^{\frac{p}{2}}|^2 dx \\ &= - \int_{\partial\Omega} |b|^{p-2} (b \cdot \nabla) \nu \cdot b d\sigma + \int_{\Omega} (b \cdot \nabla) u \cdot |b|^{p-2} b dx \\ &\leq \|\nabla \nu\|_{L^\infty} \int_{\partial\Omega} |b|^p d\sigma - \sum_i \int_{\Omega} b_i u \partial_i (|b|^{p-2} b) dx \\ &\leq C \int_{\partial\Omega} \phi^2 d\sigma + C \int_{\Omega} |u \phi \nabla \phi| dx \\ &\leq C \int_{\partial\Omega} \phi^2 d\sigma + C \|u\|_{L_w^s} \|\phi\|_{L^{\frac{2s}{s-2},2}} \|\nabla \phi\|_{L^2} \\ &\leq C \|\phi\|_{L^2} \|\phi\|_{H^1} + C \|u\|_{L_w^s} \|\phi\|_{L^2}^{1-\frac{3}{s}} \|\nabla \phi\|_{L^2}^{1+\frac{3}{s}} \\ &\leq 2 \frac{p-2}{p^2} \int_{\Omega} |\nabla \phi|^2 dx + C \|\phi\|_{L^2}^2 + C \|u\|_{L_w^s}^{\frac{2s}{s-3}} \|\phi\|_{L^2}^2, \end{aligned}$$

which yields

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \phi^2 dx + C \int_{\Omega} |\nabla \phi|^2 dx \leq C (1 + \|u\|_{L_w^s}^{\frac{2s}{s-3}}) \|\phi\|_{L^2}^2 \\ &\leq C \left(1 + \frac{\|u\|_{L_w^s}^{\frac{2s}{s-3}}}{1 + \log(e + \|u\|_{L_w^s})} \right) \|\phi\|_{L^2}^2 (1 + \log(e + \|u\|_{L_w^s})) \\ &\leq C \left(1 + \frac{\|u\|_{L_w^s}^{\frac{2s}{s-3}}}{1 + \log(e + \|u\|_{L_w^s})} \right) (1 + \log(e + y)) \|\phi\|_{L^2}^2, \end{aligned}$$

from which it follows that

$$\|b\|_{L^\infty(t_0,t;L^p)} + \int_{t_0}^t \int_{\Omega} |b|^2 |\nabla b|^2 dx d\tau \leq C (e + y(t))^{C_0\epsilon} \quad (5.7)$$

with

$$y(t) := \sup_{[t_0,t]} \|u\|_{W^{1,4}}$$

for any $0 < t_0 \leq t \leq T$ and C_0 is an absolute constant, provided that

$$\int_{t_0}^T \frac{\|u\|_{L_w^s}^{\frac{2s}{s-3}}}{1 + \log(e + \|u\|_{L_w^s})} d\tau \leq \epsilon \ll 1. \quad (5.8)$$

Testing (1.14) by u_t , using (1.12) and (1.13), we infer that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} \rho |u_t|^2 dx &= - \int_{\Omega} \rho u \cdot \nabla u \cdot u_t dx + \int_{\Omega} b \cdot \nabla b \cdot u_t dx \\ &=: I_1 + I_2. \end{aligned} \quad (5.9)$$

We first compute I_2 .

$$\begin{aligned} I_2 &= \int_{\Omega} \operatorname{div}(b \otimes b) \cdot u_t dx = - \int_{\Omega} b \otimes b : \nabla u_t dx \\ &= - \frac{d}{dt} \int_{\Omega} b \otimes b : \nabla u dx + 2 \int_{\Omega} b \otimes b_t : \nabla u dx \\ &\leq - \frac{d}{dt} \int_{\Omega} b \otimes b : \nabla u dx + C \|b_t\|_{L^2} \|b\|_{L^6} \|\nabla u\|_{L^3} \\ &\leq - \frac{d}{dt} \int_{\Omega} b \otimes b : \nabla u dx + C \|b_t\|_{L^2} \|b\|_{L^6} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|u\|_{H^2}^{\frac{1}{2}} \\ &\leq - \frac{d}{dt} \int_{\Omega} b \otimes b : \nabla u dx + \delta \|b_t\|_{L^2}^2 + \delta \|u\|_{H^2}^2 + C \|b\|_{L^6}^4 \|\nabla u\|_{L^2}^2 \end{aligned} \quad (5.10)$$

for any $0 < \delta < 1$.

We use (5.1), (5.5) and (5.6) to bound I_1 as follows.

$$\begin{aligned} I_1 &\leq \|\sqrt{\rho} u_t\|_{L^2} \|\sqrt{\rho}\|_{L^\infty} \|u\|_{L_w^s} \|\nabla u\|_{L^{\frac{2s}{s-2},2}} \\ &\leq C \|\sqrt{\rho} u_t\|_{L^2} \|u\|_{L_w^s} \|\nabla u\|_{L^2}^{1-\frac{3}{s}} \|u\|_{H^2}^{\frac{3}{s}} \\ &\leq \delta \|\sqrt{u_t}\|_{L^2}^2 + \delta \|u\|_{H^2}^2 + C \|u\|_{L_w^s}^{\frac{2s}{s-3}} \|\nabla u\|_{L^2}^2 \end{aligned} \quad (5.11)$$

for any $0 < \delta < 1$.

On the other hand, by the H^2 -theory of the Stokes system, and using (5.1), (5.5) and (5.6), we obtain

$$\begin{aligned} \|u\|_{H^2} &\leq C \left\| -\Delta u + \nabla \left(\pi + \frac{1}{2} |b|^2 \right) \right\|_{L^2} \\ &\leq C \|\rho \partial_t u + \rho u \cdot \nabla u - b \cdot \nabla b\|_{L^2} \\ &\leq C \|\sqrt{\rho} u_t\|_{L^2} + C \|u\|_{L_w^s} \|\nabla u\|_{L^{\frac{2s}{s-2},2}} + C \|b \cdot \nabla b\|_{L^2} \\ &\leq C \|\sqrt{\rho} u_t\|_{L^2} + C \|u\|_{L_w^s} \|\nabla u\|_{L^2}^{1-\frac{3}{s}} \|u\|_{H^2}^{\frac{3}{s}} + C \|b \cdot \nabla b\|_{L^2}, \end{aligned}$$

which gives

$$\|u\|_{H^2} \leq C \|\sqrt{\rho} u_t\|_{L^2} + C \|b \cdot \nabla b\|_{L^2} + C \|u\|_{L_w^s}^{\frac{s}{s-3}} \|\nabla u\|_{L^2}. \quad (5.12)$$

Testing (1.15) by $b_t - \Delta b$, using (5.5) and (5.6), we deduce that

$$\begin{aligned}
& \frac{d}{dt} \int_{\Omega} |\operatorname{curl} b|^2 dx + \int_{\Omega} (|b_t|^2 + |\Delta b|^2) dx \\
&= \int_{\Omega} (b \cdot \nabla u - u \cdot \nabla b)(b_t - \Delta b) dx \\
&\leq (\|u\|_{L_w^s} \|\nabla b\|_{L^{\frac{2s}{s-2},2}} + \|b\|_{L^6} \|\nabla u\|_{L^3}) (\|b_t\|_{L^2} + \|\Delta b\|_{L^2}) \\
&\leq C(\|u\|_{L_w^s} \|\nabla b\|_{L^2}^{1-\frac{3}{s}} \|b\|_{H^2}^{\frac{3}{s}} + C\|b\|_{L^6} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|u\|_{H^2}^{\frac{1}{2}}) (\|b_t\|_{L^2} + \|\Delta b\|_{L^2}) \\
&\leq \frac{1}{2} (\|b_t\|_{L^2}^2 + \|\Delta b\|_{L^2}^2) + \delta \|u\|_{H^2}^2 + C\|b\|_{L^6}^4 \|\nabla u\|_{L^2}^2 + C\|u\|_{L_w^{\frac{2s}{s-3}}}^{\frac{2s}{s-3}} \|\nabla b\|_{L^2}^2 + C \quad (5.13)
\end{aligned}$$

for any $0 < \delta < 1$.

It is easy to compute that

$$\begin{aligned}
\frac{d}{dt} \int_{\Omega} |b|^4 dx &\leq C \int_{\Omega} |b|^3 |b_t| dx \\
&\leq C\|b\|_{L^6}^3 \|b_t\|_{L^2} \leq \delta \|b_t\|_{L^2}^2 + C\|b\|_{L^6}^6 \quad (5.14)
\end{aligned}$$

for any $0 < \delta < 1$.

Combining (5.9), (5.10), (5.11), (5.12), (5.13) and (5.14), and taking δ small enough, we obtain

$$\begin{aligned}
& \frac{d}{dt} \int_{\Omega} (|\nabla u|^2 + |\operatorname{curl} b|^2 + b \otimes b : \nabla u + C_0 |b|^4) dx + \int_{\Omega} (\rho |u_t|^2 + |b_t|^2 + |\Delta b|^2) dx + \|u\|_{H^2}^2 \\
&\leq C\|b\|_{L^6}^4 \|\nabla u\|_{L^2}^2 + C\|u\|_{L_w^{\frac{2s}{s-3}}}^{\frac{2s}{s-3}} (\|\nabla u\|_{L^2}^2 + \|\operatorname{curl} b\|_{L^2}^2) + C\|b \cdot \nabla b\|_{L^2}^2 + C. \quad (5.15)
\end{aligned}$$

Using (5.4), (5.7), (5.8) and the Gronwall inequality, we have

$$\begin{aligned}
& \int_{\Omega} (|\nabla u|^2 + |\operatorname{curl} b|^2 + b \otimes b : \nabla u + C_0 |b|^4) dx \\
&\leq \left[\int_{\Omega} (|\nabla u_0|^2 + |\operatorname{curl} b_0|^2 + b_0 \otimes b_0 : \nabla u_0 + C_0 |b_0|^4) dx \right. \\
&\quad \left. + C\|b\|_{L^\infty(t_0, t; L^6)}^4 \int_{t_0}^t \|\nabla u\|_{L^2}^2 d\tau + C(t - t_0) + C \int_{t_0}^t \|b \cdot \nabla b\|_{L^2}^2 d\tau \right] \exp \left(\int_{t_0}^t \|u\|_{L_w^{\frac{2s}{s-3}}}^{\frac{2s}{s-3}} d\tau \right) \\
&\leq C(e + y)^{C_0 \epsilon} \exp \left[\int_{t_0}^t \frac{\|u\|_{L_w^{\frac{2s}{s-3}}}^{\frac{2s}{s-3}}}{1 + \log(e + \|u\|_{L_w^s})} d\tau (1 + \log(e + y)) \right] \\
&\leq C(e + y)^{C_0 \epsilon}. \quad (5.16)
\end{aligned}$$

Plugging (5.16) into (5.15) and integrating over $[t_0, t]$, we have

$$\int_{t_0}^t \int_{\Omega} (\rho |u_t|^2 + |b_t|^2 + |\Delta b|^2) dx d\tau + \int_{t_0}^t \|u\|_{H^2}^2 d\tau \leq C(e + y)^{C_0 \epsilon}. \quad (5.17)$$

Applying ∂_t to (1.15), testing by u_t , using (1.12) and (1.13), we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho |u_t|^2 dx + \int_{\Omega} |\nabla u_t|^2 dx \\
&= - \int_{\Omega} \rho u \cdot \nabla |u_t|^2 dx - \int_{\Omega} \rho u \cdot \nabla (u \cdot \nabla u \cdot u_t) dx \\
&\quad - \int_{\Omega} \rho u_t \cdot \nabla u \cdot u_t dx + \int_{\Omega} b \otimes b_t : \nabla u_t dx + \int_{\Omega} b_t \otimes b : \nabla u_t dx \\
&\leq C \|u\|_{L^6} \|\sqrt{\rho} u_t\|_{L^3} \|\nabla u_t\|_{L^2} + C \|u\|_{L^6} \|\nabla u\|_{L^6} \|u_t\|_{L^6} \|\nabla u\|_{L^2} \\
&\quad + C \|u\|_{L^6}^2 \|\Delta u\|_{L^2} \|u_t\|_{L^6} + C \|u\|_{L^6}^2 \|\nabla u\|_{L^6} \|\nabla u_t\|_{L^2} \\
&\quad + C \|\sqrt{\rho} u_t\|_{L^4}^2 \|\nabla u\|_{L^2} + C \|b\|_{L^6} \|b_t\|_{L^3} \|\nabla u_t\|_{L^2} \\
&\leq C \|\nabla u\|_{L^2} \|\sqrt{\rho} u_t\|_{L^2}^{\frac{1}{2}} \|\sqrt{\rho} u_t\|_{L^6}^{\frac{1}{2}} \|\nabla u_t\|_{L^2} \\
&\quad + C \|\nabla u\|_{L^2}^2 \|u\|_{H^2} \|\nabla u_t\|_{L^2} + C \|\nabla u\|_{L^2} \|\sqrt{\rho} u_t\|_{L^2}^{\frac{1}{2}} \|\sqrt{\rho} u_t\|_{L^6}^{\frac{3}{2}} \\
&\quad + C \|b\|_{L^6} \|b_t\|_{L^3} \|\nabla u_t\|_{L^2} \\
&\leq C \|\nabla u\|_{L^2} \|\sqrt{\rho} u_t\|_{L^2}^{\frac{1}{2}} \|\nabla u_t\|_{L^2}^{\frac{3}{2}} + C \|\nabla u\|_{L^2}^2 \|u\|_{H^2} \|\nabla u_t\|_{L^2} \\
&\quad + C \|\nabla u\|_{L^2} \|\sqrt{\rho} u_t\|_{L^2}^{\frac{1}{2}} \|\nabla u_t\|_{L^2}^{\frac{3}{2}} + C \|b\|_{L^6} \|b_t\|_{L^3} \|\nabla u_t\|_{L^2} \\
&\leq \frac{1}{4} \|\nabla u_t\|_{L^2}^2 + C \|\nabla u\|_{L^2}^4 (\|\sqrt{\rho} u_t\|_{L^2}^2 + \|u\|_{H^2}^2) + C \|b\|_{L^6}^2 \|b_t\|_{L^3}^2 \\
&\leq \frac{1}{4} \|\nabla u_t\|_{L^2}^2 + C \|\nabla u\|_{L^2}^4 (\|\sqrt{\rho} u_t\|_{L^2}^2 + \|u\|_{H^2}^2) + \frac{1}{4} \|\operatorname{curl} b_t\|_{L^2}^2 + C \|b\|_{L^6}^4 \|b_t\|_{L^2}^2. \quad (5.18)
\end{aligned}$$

Applying ∂_t to (1.15), testing by b_t and using (1.12), we get

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_{\Omega} |b_t|^2 dx + \int_{\Omega} |\operatorname{curl} b_t|^2 dx \\
&= - \int_{\Omega} (u_t \cdot \nabla b - b_t \nabla u - b \cdot \nabla u_t) b_t dx \\
&\leq \|u_t\|_{L^6} \|\nabla b\|_{L^2} \|b_t\|_{L^3} + \|\nabla u\|_{L^2} \|b_t\|_{L^4}^2 + \|\nabla u_t\|_{L^2} \|b\|_{L^6} \|b_t\|_{L^3} \\
&\leq \frac{1}{4} \|\nabla u_t\|_{L^2}^2 + \frac{1}{4} \|\operatorname{curl} b_t\|_{L^2}^2 + C \|\nabla b\|_{L^2}^4 \|b_t\|_{L^2}^2 + C \|\nabla u\|_{L^2}^4 \|b_t\|_{L^2}^2. \quad (5.19)
\end{aligned}$$

Combining (5.18) and (5.19) and integrating over $[t_0, t]$, we have

$$\int_{\Omega} (|\rho u_t|^2 + |b_t|^2) dx + \int_{t_0}^t \int_{\Omega} (|\nabla u_t|^2 + |\operatorname{curl} b_t|^2) dx d\tau \leq C(e + y)^{C_0 \epsilon}. \quad (5.20)$$

Similarly to (5.12), we deduce that

$$\begin{aligned}
\|u\|_{H^2} &\leq C \|\sqrt{\rho} u_t\|_{L^2} + C \|u\|_{L^6} \|\nabla u\|_{L^3} + C \|b\|_{L^6} \|\nabla b\|_{L^3} \\
&\leq C \|\sqrt{\rho} u_t\|_{L^2} + C \|u\|_{L^6} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|u\|_{H^2}^{\frac{1}{2}} + C \|b\|_{L^6} \|\nabla b\|_{L^2}^{\frac{1}{2}} \|b\|_{H^2}^{\frac{1}{2}},
\end{aligned}$$

which leads to

$$\|u\|_{H^2}^2 \leq C\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^6 + C\|\nabla b\|_{L^2}^6 + \frac{1}{2}\|b\|_{H^2}^2. \quad (5.21)$$

Similarly, we have

$$\begin{aligned} \|b\|_{H^2} &\leq C\|b_t + u \cdot \nabla b - b \cdot \nabla u\|_{L^2} \\ &\leq C\|b_t\|_{L^2} + C\|u\|_{L^6}\|\nabla b\|_{L^3} + C\|b\|_{L^6}\|\nabla u\|_{L^3} \\ &\leq C\|b_t\|_{L^2} + C\|u\|_{L^6}\|\nabla b\|_{L^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{1}{2}} + C\|b\|_{L^6}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|u\|_{H^2}^{\frac{1}{2}}, \end{aligned}$$

which implies

$$\|b\|_{H^2}^2 \leq C\|b_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^6 + C\|\nabla b\|_{L^2}^6 + \frac{1}{2}\|u\|_{H^2}^2. \quad (5.22)$$

Combining (5.21) and (5.22), using (5.20) and (5.16), we conclude that

$$\|u\|_{H^2}^2 + \|b\|_{H^2}^2 \leq C(e + y)^{C_0\epsilon}, \quad (5.23)$$

and thus

$$\|u\|_{L^\infty(0,T;H^2)} + \|b\|_{L^\infty(0,T;H^2)} \leq C. \quad (5.24)$$

Now it is standard to prove that

$$\|u\|_{L^2(0,T;H^3)} + \|b\|_{L^2(0,T;H^3)} \leq C, \quad (5.25)$$

$$\|\rho\|_{L^\infty(0,T;H^2)} \leq C. \quad (5.26)$$

(II) Let (1.22) hold true.

Similarly to (5.7), we take $s = \infty$ and using (2.4), we still get (5.7) provided that

$$\int_{t_0}^T \|u(t)\|_{BMO}^2 dt \leq \epsilon \ll 1. \quad (5.27)$$

We still have (5.9), (5.10), (5.11) with $s = \infty$, (5.12) with $s = \infty$, (5.13) with $s = \infty$, (5.14), (5.15) with $s = \infty$, and then using (5.27) and (2.4), we arrive at (5.16) and (5.17). Then by the same calculations as that in (5.18)-(5.26), we conclude that (5.18)-(5.26) hold true.

This completes the proof of Theorem 1.3. □

6 Proof of Theorem 1.4

This section is devoted to the proof of Theorem 1.4, we only need to establish a priori estimates.

First, using the formula $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$, and the fact that $|d| = 1$ implies $d\Delta d = -|\nabla d|^2$, we have the following equivalent equation

$$\frac{1}{2}d_t - \frac{1}{2}d \times d_t = \Delta d + d|\nabla d|^2. \quad (6.1)$$

Testing (6.1) by d_t and using $(a \times b) \cdot b = 0$ and $d \cdot d_t = 0$, we get

$$\frac{d}{dt} \int_{\Omega} |\nabla d|^2 dx + \int_{\Omega} |d_t|^2 dx \leq 0. \quad (6.2)$$

Testing (1.23) by $-\Delta d_t$ and using $|d| = 1$, we find that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\Delta d|^2 dx + \int_{\Omega} |\nabla d_t|^2 dx = - \int_{\Omega} (d|\nabla d|^2 + d \times \Delta d) \cdot \Delta d_t dx \\ &= \int_{\Omega} \nabla (d|\nabla d|^2 + d \times \Delta d) \cdot \nabla d_t dx \\ &\leq C(\|\nabla d\|_{L^q} \|\nabla d\|_{L^{\frac{4q}{q-2}}}^2 + \|\nabla d\|_{L^q} \|\Delta d\|_{L^{\frac{2q}{q-2}}} + \|\nabla \Delta d\|_{L^2}) \|\nabla d_t\|_{L^2} \\ &\leq C(\|\nabla d\|_{L^q} \|\Delta d\|_{L^{\frac{2q}{q-2}}} + \|\nabla \Delta d\|_{L^2}) \|\nabla d_t\|_{L^2} \\ &\leq C(\|\nabla d\|_{L^q} \|\Delta d\|_{L^2}^{1-\frac{3}{q}} \|d\|_{H^3}^{\frac{3}{q}} + \|d\|_{H^3}) \|\nabla d_t\|_{L^2} \\ &\leq \frac{1}{4} \|\nabla d_t\|_{L^2}^2 + \delta \|d\|_{H^3}^2 + C \|\nabla d\|_{L^q}^{\frac{2q}{q-3}} \|\Delta d\|_{L^2}^2 \end{aligned} \quad (6.3)$$

for any $0 < \delta < 1$. Here we have used the Gagliardo-Nirenberg inequalities:

$$\|\nabla d\|_{L^{\frac{4q}{q-2}}}^2 \leq C \|d\|_{L^\infty} \|\Delta d\|_{L^{\frac{2q}{q-2}}}, \quad (6.4)$$

$$\|\Delta d\|_{L^{\frac{2q}{q-2}}} \leq C \|\Delta d\|_{L^2}^{1-\frac{3}{q}} \|d\|_{H^3}^{\frac{3}{q}}. \quad (6.5)$$

Applying ∂_i to (1.23), we get

$$\partial_i d_t - \Delta \partial_i d = \partial_i (d|\nabla d|^2) + \partial_i d \times \Delta d + d \times \Delta \partial_i d.$$

Testing the above equation by $\Delta \partial_i d$, summing over i , and using (6.4) and (6.5) and $|d| = 1$, we obtain

$$\begin{aligned} \|d\|_{H^3} &\leq C(\|d\|_{L^2} + \|\nabla \Delta d\|_{L^2}) \\ &\leq C + C \|\nabla d_t\|_{L^2} + C \|\nabla (d|\nabla d|^2)\|_{L^2} + \sum_i C \|\partial_i d \times \Delta d\|_{L^2} \\ &\leq C + C \|\nabla d_t\|_{L^2} + C \|\nabla d\|_{L^q} \|\nabla d\|_{L^{\frac{4q}{q-2}}}^2 + C \|\nabla d\|_{L^q} \|\Delta d\|_{L^{\frac{2q}{q-2}}} \\ &\leq C + C \|\nabla d_t\|_{L^2} + C \|\nabla d\|_{L^q} \|\Delta d\|_{L^{\frac{2q}{q-2}}} \\ &\leq C + C \|\nabla d_t\|_{L^2} + C \|\nabla d\|_{L^q} \|\Delta d\|_{L^2}^{1-\frac{3}{q}} \|d\|_{H^3}^{\frac{3}{q}}, \end{aligned}$$

which yields

$$\|d\|_{H^3} \leq C + C\|\nabla d_t\|_{L^2} + C\|\nabla d\|_{L^q}^{\frac{q}{q-3}} \|\Delta d\|_{L^2}. \quad (6.6)$$

Plugging (6.6) into (6.3) and taking δ small enough, we have

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} |\Delta d|^2 dx + \int_{\Omega} |\nabla d_t|^2 dx \\ & \leq C + C\|\nabla d\|_{L^q}^{\frac{2q}{q-3}} \|\Delta d\|_{L^2}^2 \\ & \leq C + C \frac{\|\nabla d\|_{L^q}^{\frac{2q}{q-3}}}{1 + \log(e + \|\nabla d\|_{L^q})} \|\Delta d\|_{L^2}^2 \log(e + \|\nabla d\|_{L^q}) \\ & \leq C + C \frac{\|\nabla d\|_{L^q}^{\frac{2q}{q-3}}}{1 + \log(e + \|\nabla d\|_{L^q})} \|\Delta d\|_{L^2}^2 \log(e + y), \end{aligned}$$

which implies

$$\int_{\Omega} |\Delta d|^2 dx + \int_{t_0}^t \int_{\Omega} |\nabla d_t|^2 dx d\tau \leq C(e + y)^{C_0\epsilon}. \quad (6.7)$$

Provided that

$$\int_{t_0}^T \frac{\|\nabla d\|_{L^q}^{\frac{2q}{q-3}}}{1 + \log(e + \|\nabla d\|_{L^q})} d\tau \leq \epsilon \ll 1,$$

with $y(t) := \sup_{[t_0, t]} \|d\|_{H^3}$ for any $0 < t_0 \leq t \leq T$ and C_0 is an absolute constant.

It follows from (1.23), (6.6) and (6.7) that

$$\int_{\Omega} |d_t|^2 dx + \int_{t_0}^t \|d\|_{H^3}^2 d\tau \leq C(e + y)^{C_0\epsilon}. \quad (6.8)$$

Applying ∂_t to (1.23), testing by $-\Delta d_t$, and using $|d| = 1$, (6.7) and (6.8), we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla d_t|^2 dx + \int_{\Omega} |\Delta d_t|^2 dx = - \int_{\Omega} [\partial_t(d|\nabla d|^2) + d_t \times \Delta d] \Delta d_t dx \\ & \leq C(\|\nabla d\|_{L^6}^2 \|d_t\|_{L^6} + \|\nabla d\|_{L^6} \|\nabla d_t\|_{L^3} + \|d_t\|_{L^\infty} \|\Delta d\|_{L^2}) \|\Delta d_t\|_{L^2} \\ & \leq C(\|\nabla d\|_{L^6}^2 \|d_t\|_{L^6} + \|\Delta d\|_{L^2} \|\nabla d_t\|_{L^2}^{\frac{1}{2}} \|\Delta d_t\|_{L^2}^{\frac{1}{2}} + \|\Delta d\|_{L^2} \|d_t\|_{L^2}) \|\Delta d_t\|_{L^2} \\ & \leq \frac{1}{2} \|\Delta d_t\|_{L^2}^2 + C\|d\|_{H^2}^4 \|d_t\|_{H^1}^2 + C\|d\|_{H^2}^2 \|d_t\|_{L^2}^2, \end{aligned}$$

which implies

$$\int_{\Omega} |\nabla d_t|^2 dx + \int_{t_0}^t \|\Delta d_t\|_{L^2}^2 d\tau \leq C(e + y)^{C_0\epsilon}. \quad (6.9)$$

It follows from (6.6), (6.7), (6.8) and (6.9) that

$$\|d\|_{H^3} \leq C + C\|\nabla d_t\|_{L^2} + C\|\nabla d\|_{L^6}^2 \|\Delta d\|_{L^2} \leq C(e + y)^{C_0\epsilon},$$

which leads to

$$\|d\|_{L^\infty(0,T;H^3)} \leq C.$$

This completes the proof of Theorem 1.4. □

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